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INVARIANT HYPERPLANES AND DARBOUX INTEGRABILITY FOR d-DIMENSIONAL POLYNOMIAL DIFFERENTIAL SYSTEMS (*)

BY

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ABSTRACT. – For a class of polynomial differential systems of degree $(m_1, \ldots, m_d)$ in $\mathbf{R}^d$ which is open and dense in the set of all polynomial differential systems of degree $(m_1, \ldots, m_d)$ in $\mathbf{R}^d$, we study the maximal number of invariant hyperplanes. This is a well known problem in dimension $d = 2$ (see for instance [1,12,16]). Furthermore, using the Darboux theory of integrability we analyse when can be possible to find a first integral of a polynomial vector field of degree $(m_1, \ldots, m_d)$ in $\mathbf{R}^d$ by knowing the existence of a sufficient number of invariant hyperplanes. © 2000 Éditions scientifiques et médicales Elsevier SAS

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1. Introduction

By definition a *polynomial system* in $\mathbf{R}^d$ is a differential system of the form:

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = P_i(x_1, \ldots, x_d), \quad i = 1, \ldots, d, \tag{1}$$

where the polynomial P_i and the independent variable t (usually called the *time*) are real. If $m_i = \deg P_i$ we say that $\mathbf{m} = (m_1, \ldots, m_d)$ is the degree of the polynomial system. In the rest of the paper without loss of generality we can assume that $m_1 \geqslant \cdots \geqslant m_d$.

We denote by

$$D = \sum_{i=1}^{d} P_i \frac{\partial}{\partial x_i} \tag{2}$$

the *differential operator* (also called the *vector field*) associated to system (1). Let U be an open subset of $\mathbf{R}^d$. Here a nonconstant analytic function $H : U \to \mathbf{R}$ is called a *first integral* of the system in U if it is constant on all solution curves $(x_1(t), \ldots, x_d(t))$ of system (1) contained in U; i.e., $H(x_1(t), \ldots, x_d(t)) =$ constant for all values of t for which the solution $(x_1(t), \ldots, x_d(t))$ is defined on U. Clearly H is a first integral of system (1) on U if and only if $DH \equiv 0$ on U.

The first integrals $H_1, \ldots, H_s$ are *independent* in U if

$$\operatorname{Rank}\left(\frac{\partial(H_1, \ldots, H_s)}{\partial(x_1, \ldots, x_s)}(x)\right) = s.$$

If $H_1, \ldots, H_s$ are independents, then every trayectory through $\overline{x} \in U$ is contained in the $(d - s)$-dimensional manifold defined by

$$H_1(x) = H_1(\overline{x}), \ldots, H_s(x) = H_s(\overline{x}). \tag{3}$$

If $s = d - 1$ then we say that the system is *integrable*. In this case, the trayectory through the point $\overline{x} \in U$ is defined by (3).

An *invariant algebraic variety* of system (1) is an algebraic variety $f(x_1, \ldots, x_d) = 0$ with f belonging to the ring of polynomials in the variables $x_1, \ldots, x_d$ with coefficients in $\mathbf{R}$ (i.e., $\mathbf{R}[x_1, \ldots, x_d]$), such that for some polynomial $K \in \mathbf{R}[x_1, \ldots, x_d]$ we have $Df = Kf$. Therefore, if a solution curve of system (1) has a point on the algebraic variety $f = 0$, then the whole solution curve is contained in $f = 0$. The polynomial K is called the *cofactor* of the invariant algebraic variety $f = 0$. We remark that if the polynomial system has degree $\mathbf{m} = (m_1, \ldots, m_d)$, with $m_1 \geqslant m_2 \geqslant \cdots \geqslant m_d$, then any cofactor has at most degree $m_1 - 1$. If the degree of f is 1 then we say that $f = 0$ is an *invariant hyperplane*.

The algebraic feature of polynomial systems renders natural certain algebro–geometric questions as the following two. Recognize when a polynomial system (1) has invariant algebraic varieties, or has a first integral associated to these invariant algebraic varieties. We deal with both questions with more emphasis on the invariant algebraic varieties of degree 1 (i.e., on the invariant hyperplanes). The study of these questions started essentially with Darboux [7] and Poincaré [11] in the plane. Thus, already in 1878, Darboux showed how the first integrals of planar polynomial systems possessing sufficient algebraic solutions are constructed, and these results were extended to d-dimensional polynomial systems by Jouanolou [9] in 1979, see also [15]. See [5,6] for some new improvements to the Darboux theory of integrability which essentially take into account the exponential factors.

In Section 2, we introduce the part of the Darboux theory of integrability in arbitrary dimension that we shall need, i.e. the part that only uses the invariant algebraic varieties for constructing a first integral, see Section 2 and Theorem 2 for more details.

In Section 3, we show how to use the results of Section 2 for finding some first integrals of the 3-dimensional Lotka–Volterra systems. These systems introduced by Lotka [10] and Volterra [14] appear in ecology where they model three species in competition, and they have been widely used in applied mathematics and in a big variety of problems in physics, see for more details [3]. In fact, in Section 2 we compute all the invariant planes of the 3-dimensional Lotka–Volterra systems (see Lemma 3), and apply them to compute first integrals of these systems (see Proposition 4 and Corollary 5). Improvements of the results of this section can be find in [3].

The main results of this paper are in Section 5, there we study how many invariant hyperplanes can have a polynomial system of degree $\mathbf{m} = (m_1, \ldots, m_d)$ for the subclass of regular polynomial systems in $\mathbf{R}^d$ (see Section 4), this subclass is open and dense inside the class of all polynomial systems in $\mathbf{R}^d$. Given the degreee $\mathbf{m} = (m_1, \ldots, m_d)$ of the polynomial system and consequently the dimension d where the system is defined, we also study in function of $\mathbf{m}$ and d when can exist a sufficiently larger number of invariant hyperplanes that force the existence of a first integral for the system. All these results are summarized in Theorems 9 and 10.

2. Darboux theory of integrability in dimension d

In this section we present the part of the Darboux theory on integrability which tell us how to construct a first integral for a d-dimensional polynomial system using his invariant algebraic varieties. Since its proof is very easy, we do it. First we recall the following well known result.

LEMMA 1. – *The real vector space $\mathbf{R}_m[x_1,\dots,x_d]$, formed by all polynomials of $\mathbf{R}[x_1,\dots,x_d]$ of degree at most m, has dimension*

$$\Delta(d,m)=\binom{d+m}{m}.$$

For a polynomial system of degree $\mathbf{m}=(m_1,\dots,m_d)$ we write $\Delta(d;\mathbf{m})=\Delta(d;m_1)$.

The Darboux theory of integrability restricted to the use of invariant algebraic varieties is summarized into the following theorem. More general results using additionaly independent singular points and exponential factors see [4–6].

THEOREM 2. – *Suppose that a polynomial system* (1) *of degree* $\mathbf{m}$ *in* $\mathbf{R}^d$ *admits q invariant algebraic varieties $f_i=0$ with cofactors K_i for $i=1,\dots,q$.*

(a) *If there exist $\lambda_i\in\mathbf{R}$ not all zero such that $\sum_{i=1}^q\lambda_iK_i=0$, then $|f_1|^{\lambda_1}\cdots|f_q|^{\lambda_q}$ is a first integral of the system.*
(b) *If $q=\Delta(d,m_1-1)+1$, then there exist $\lambda_i\in\mathbf{R}$ not all zero such that $\sum_{i=1}^q\lambda_iK_i=0$.*

Proof. – From

$$\begin{aligned}D\big(|f_1|^{\lambda_1}\cdots|f_q|^{\lambda_q}\big)&=\pm\big(|f_1|^{\lambda_1}\cdots|f_q|^{\lambda_q}\big)\left(\sum_{i=1}^q\lambda_i\frac{Df_i}{f_i}\right)\\&=\pm\big(|f_1|^{\lambda_1}\cdots|f_q|^{\lambda_q}\big)\left(\sum_{i=1}^q\lambda_iK_i\right)\equiv 0,\end{aligned}$$

statement (a) follows.

Suppose that $q = \Delta(d, m_1 - 1) + 1 = \dim \mathbf{R}_{m_1-1}[x_1, \dots, x_d] + 1$, see Lemma 1. Since $K_i \in \mathbf{R}_{m_1-1}[x_1, \dots, x_d]$ for $i = 1, \dots, q$, the polynomials K_i must be linearly dependent on $\mathbf{R}_{m_1-1}[x_1, \dots, x_d]$. Hence, there are $\lambda_i \in \mathbf{R}$ not all zero such that $\sum_{i=1}^{q} \lambda_i K_i = 0$. Consequently, from (a) statement (b) holds. □

3. Darboux integrability for the 3-dimensional Lotka–Volterra system

As an application of the Darboux theory of integrability in dimension larger than 2, we obtain some first integrals for the 3-dimensional Lotka–Volterra system

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = x_i\left(\sum_{j=1}^{3} a_{ij}x_j + \lambda_i\right), \quad i = 1, 2, 3. \tag{4}$$

Here we assume that $a_{ii} = 0$. Then, after a change of variables system (4) becomes

$$\begin{aligned} &\frac{\mathrm{d}x}{\mathrm{d}t} = x(Cy + z + \lambda), \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = y(x + Az + \mu),\\ &\frac{\mathrm{d}z}{\mathrm{d}t} = z(Bx + y + \nu). \end{aligned} \tag{5}$$

If $\lambda = \mu = \nu \neq 0$ the change of variables $(x, y, z, t) \to (\overline{x}, \overline{y}, \overline{z}, \overline{t})$ given by

$$\overline{x} = x\,\mathrm{e}^{-\lambda t}, \qquad \overline{y} = y\,\mathrm{e}^{-\lambda t}, \qquad \overline{z} = z\,\mathrm{e}^{-\lambda t}, \qquad \overline{t} = \frac{1}{\lambda}\,\mathrm{e}^{\lambda t},$$

transforms system (5) in the same with $\lambda = \mu = \nu = 0$. Therefore, notice that to study the dynamics of the 3-dimensional Lotka–Volterra systems with $\lambda = \mu = \nu \neq 0$ is equivalent to study the dynamics of the same system with $\lambda = \mu = \nu = 0$.

These systems have been studied by several authors, see for instance [8,3] and the references inside these papers.

The next result is about the existence of invariant planes, together with their cofactors, and the conditions for their existence. But before we associate to a given 3-dimensional Lotka–Volterra system (5) two *equivalent* 3-dimensional Lotka–Volterra systems, doing circular permutation of the variables x, y, z and of the parameters λ, μ, ν and A, B, C.

LEMMA 3. – *All invariant planes of system* (5), *modulo equivalences, are the following ones*:

(a) $f = x = 0$ *with cofactor* $K = Cy + z + \lambda$;
(b) $f = x - Cy = 0$ *with cofactor* $K = z + \lambda$ *if and only if* $A = 1$, $C \neq 0$ *and* $\lambda = \mu$;
(c) $f = x - Cy + ACz = 0$ *with cofactor* $K = \lambda$ *if and only if* $1 + ABC = 0$ *and* $\lambda = \mu = \nu$. *If* $\lambda = \mu = \nu = 0$, *then system* (5) *has the first integral* $f = x - Cy + ACz$.

Proof. – The proof is obtained finding the linear f's satisfying equation $Df = Kf$. The second part of statement (c) follows from the fact that in this case, $K = 0$, and consequently f is a first integral. □

The next proposition exhibits for systems (5) the first integrals obtained from Theorem 2 by using only the invariant planes given in Lemma 3.

PROPOSITION 4. – *A relation of the Darboux first integrals for the Lotka–Volterra system* (5), *modulo equivalencies, obtained only using invariant planes is the following*:

(a) $H = x - Cy + ACz$ *if* $1 + ABC = 0$ *and* $\lambda = \mu = \nu = 0$;
(b) $H = |x|^{AB}|y|^{-B}z$ *if* $1 + ABC = 0$ *and* $\lambda = C(\nu - B\mu)$;
(c) $H = x^{-1}|y|^{-BC}|z|^{C}|x - Cy|^{BC+1}$ *if* $A = 1$, $C \neq 0$, $\lambda = \mu$ *and* $\nu = 0$;
(d) $H = x|y|^{r}|z|^{-C}|x - Cy|^{-(1+r)}|y - z|^{C-r}$ *for all* $r \in \mathbf{R}$ *if* $A = B = 1$, $C \neq 0$ *and* $\lambda = \mu = \nu = 0$.

Moreover, every Darboux first integral obtained using only invariant planes is a function of at most two of the previous first integrals.

Proof. – Statement (a) follows from Lemma 3(c). For proving the remainder statements we need to introduce all the invariant planes with their respective cofactors:

$f_1 = x = 0$ with $K_1 = Cy + z + \lambda$;
$f_2 = y = 0$ with $K_2 = x + Az + \mu$;

$f_3 = z = 0$ with $K_3 = Bx + y + \nu$;
$f_4 = x - Cy = 0$ with $K_4 = z + \lambda$ if and only if $A = 1$, $C \neq 0$ and $\lambda = \mu$;
$f_5 = y - Az = 0$ with $K_5 = x + \mu$ if and only if $B = 1$, $A \neq 0$ and $\mu = \nu$;
$f_6 = z - Bx = 0$ with $K_6 = y + \lambda$ if and only if $C = 1$, $B \neq 0$ and $\lambda = \nu$;
$f_7 = x - Cy + ACz = 0$ with $K_7 = \lambda$ if and only if $1 + ABC = 0$.

We have omitted in this relation the two circular permuted invariant planes associated to $f_7 = 0$, because taking into account $1 + ABC = 0$ they concide with $f_7 = 0$.

We use Theorem 2(a) for finding the first integrals of system (5), therefore we need to compute what linear combinations of the cofactors $K_1, \ldots, K_7$ are dependent. Thus, if $\alpha_{i_1}, \ldots, \alpha_{i_r}$ are nonzero real numbers such that $\alpha_{i_1} K_{i_1} + \cdots + \alpha_{i_r} K_{i_r} = 0$, then $H = |f_{i_1}|^{\alpha_{i_1}} \cdots |f_{i_r}|^{\alpha_{i_r}}$ is a first integral.

Now easy but tedious computations allow to prove statements (b)–(d). We have omitted all the first integral which are function of the ones that appear in statements (a)–(d). □

COROLLARY 5. – *The Lotka–Volterra system* (5) *it is Darboux integrable using only invariant planes in the following cases*:

(a) $\lambda = \mu = \nu = 0$ *and* $1 + ABC = 0$ *with the first integrals of Theorem* 4(a) *and* (b).
(b) $\lambda = \mu = \nu = 0$, $A = B = 1$ *and* $C \neq 0$ *with the first integrals of Theorem* 4(d).
(c) $\lambda = \mu, \nu = 0$, $A = 1$ *and* $1 + BC = 0$ *with the first integrals of Theorem* 4(b) *and* (c).

Proof. – It follows easily checking that two of the first integrals in each statement are linearly independent. We note that in statement (d) of Theorem 4 we have different integrals according to the different values of r. □

4. Regular polynomial systems

In what follows we denote by $P = (P_1, \ldots, P_d): \mathbf{R}^d \to \mathbf{R}^d$ the real polynomial vector field associated to polynomial system (1), and by D

the differential operator defined in (2). If $f \in \mathbf{R}[x_1, \dots, x_d]$, we define $D^0 f = f$ and $D^n f = D(D^{n-1} f)$ for all integer $n \geqslant 1$.

To each polynomial vector field P we associate the following $(d-1) \times d$ matrix

$$M_P = \begin{pmatrix} P_1 & \cdots & P_d \\ DP_1 & \cdots & DP_d \\ \vdots & \ddots & \vdots \\ D^{d-2}P_1 & \cdots & D^{d-2}P_d \end{pmatrix},$$

where the kth file $(D^{k-1}P_1, \dots, D^{k-1}P_d)$ is denoted by $D^{k-1}P$. We define the polynomial vector field $B_P : \mathbf{R}^d \to \mathbf{R}^d$ associated to P as $B_P = (B_1, \dots, B_d)$ where

$$B_i = (-1)^{i+1} \begin{vmatrix} P_1 & \cdots & P_{i-1} & P_{i+1} & \cdots & P_d \\ DP_1 & \cdots & DP_{i-1} & DP_{i+1} & \cdots & DP_d \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ D^{d-2}P_1 & \cdots & D^{d-2}P_{i-1} & D^{d-2}P_{i+1} & \cdots & D^{d-2}P_d \end{vmatrix},$$

for $i = 1, \dots, d$.

Let $\mathbf{m} = (m_1, \cdots, m_d)$ be with $m_1 \geqslant m_2 \geqslant \cdots \geqslant m_d$. We define the class of *regular polynomial systems or vector fields of degree* $\mathbf{m}$ *in* $\mathbf{R}^d$, and we denote it by $\mathcal{R}(d, \mathbf{m})$, as follows: $P \in \mathcal{R}(d, \mathbf{m})$ if and only if

($\mathcal{R}_1$) P is a polynomial vector field of degree $\mathbf{m}$ in $\mathbf{R}^d$;

($\mathcal{R}_2$) P has finitely many invariant hyperplanes;

($\mathcal{R}_3$) on each invariant hyperplane π of P the set of singular points of the vector field B_P is an algebraic variety of codimension at most 1 in π.

Let $\varphi(t)$ be a solution curve of the vector field P. Then it is easy to verify that the kth derivative of $\varphi(t)$ with respect to t is given by

$$\varphi^{(k)}(t) = D^{k-1}P\big(\varphi(t)\big), \tag{6}$$

for $k = 1, 2, \dots$.

We say that the curve $\varphi(t)$ in $\mathbf{R}^d$ is *regular* at $t = t_0$ if the $d-1$ vectors $\varphi'(t_0), \dots, \varphi^{(d-1)}(t_0)$ are linearly independent in $\mathbf{R}^d$ (when $d = 3$

the notion of regularity coincides with the usual definition of biregularity for a curve $\varphi(t)$ in $\mathbf{R}^3$; see for instance [2, p. 285]). This fact motivates the definition of the class of regular polynomial vector fields, because from (6), it follows that a solution curve is regular at $t = t_0$ if and only if $B_P(\varphi(t_0)) = (B_1(\varphi(t_0)), \ldots, B_d(\varphi(t_0)))$ is not the zero vector.

The following are examples of nonregular polynomial vector fields $(P_1, \ldots, P_d)$ that will justify the hypothesis of Theorems 8 and 9 as well as the independence between conditions $\mathcal{R}_2$ and $\mathcal{R}_3$.

Example 1. – If $P_1, \ldots, P_{d-2}$ are arbitrary polynomials, and $P_{d-1} \equiv P_d \equiv 0$, then the polynomial vector field $P = (P_1, \ldots, P_d)$ of $\mathbf{R}^d$ satisfies neither $\mathcal{R}_2$ and $\mathcal{R}_3$; because the hyperplanes $x_{d-1} =$ constant and $x_d =$ constant are invariant, and the vector field B_P is identically zero, because the last two columns of the matrix M_P are identically zero.

Example 2. – If

$$P_1 = \frac{1}{2}x_1^2 + 1, \qquad P_2 = x_2^2 + 1, \qquad P_3 \equiv 0,$$

then the vector field $P = (P_1, P_2, P_3)$ of $\mathbf{R}^3$ does not verify $\mathcal{R}_2$, because the planes $x_3 =$ costant are invariant, but P satisfies $\mathcal{R}_3$ because $B_P = (0, 0, (\frac{1}{2}x_1^2 + 1)(x_2^2 + 1)(2x_2 - x_1))$ and the plane $2x_2 - x_1 = 0$ is not invariant.

Example 3. – If $P_1 = x_3(x_1 - 1)$, $P_2 = x_2 - 2$ and $P_3 = x_3$, then the vector field $P = (P_1, P_2, P_3)$ of $\mathbf{R}^3$ verifies $\mathcal{R}_2$ but not $\mathcal{R}_3$, because the only invariant planes of P are $x_1 = 1$, $x_2 = 2$ and $x_3 = 0$; and $B_P = (0, (x_1 - 1)(x_1 + x_3 - 2)x_3^2, (x_1 - 1)(x_2 - 2)(2 - x_1 - x_3)x_3)$ is identically zero on $x_1 = 1$ and $x_3 = 0$.

By using the standard techniques (see for instance [13]) it is not difficult to see that the subclass of regular polynomial vector fields of degree $\mathbf{m}$ in $\mathbf{R}^d$ is an open and dense set inside the class of all polynomial vector fields of degree $\mathbf{m}$ in $\mathbf{R}^d$. In what follows we shall study the maximal number of invariant hyperplanes for the class of regular polynomial vector fields.

5. Invariant hyperplanes and integrability for regular polynomial systems

Let $P \in \mathcal{R}(d, \mathbf{m})$. We denote by $\alpha(d, \mathbf{m}, P)$ the number of invariant hyperplanes of P, and by $\alpha(d, \mathbf{m})$ the supremum of $\alpha(d, \mathbf{m}, P)$ when P varies in $\mathcal{R}(d, \mathbf{m})$.

The next result gives an upper bound for $\alpha(d, \mathbf{m})$; i.e., for the number of maximal invariant hyperplanes that a regular polynomial system of degree $\mathbf{m}$ in $\mathbf{R}^d$ can have.

PROPOSITION 6. – *The next inequality holds*

$$\alpha(d, \mathbf{m}) \leqslant \left(\sum_{i=1}^{d} m_i\right) + (m_1 - 1)\binom{d}{2} = \beta(d, \mathbf{m}).$$

Proof. – Let $P \in \mathcal{R}(d, \mathbf{m})$. From the assumptions of regularity there exists a straight line r such that it intersects each invariant hyperplane of P in a unique point which is not singular for the vector field B_P. Without loss of generality we choose the coordinates in such a way that r coincides with the x_d-axis (i.e., r has equation $x_1 = \cdots = x_d = 0$). Therefore, any invariant hyperplane π can be written as

$$x_d = A_1 x_1 + \cdots + A_{d-1} x_{d-1} + A_d. \tag{7}$$

Thus $r \cap \pi$ is the point $(0, \ldots, 0, A_d)$.

Let $\varphi(t)$ be the solution curve of the vector field P defined on an open interval I containing the 0 such that $\varphi(0) = (0, \ldots, 0, A_d)$. Since $\varphi(0)$ is not a singular point of B_P, from Remark 1 and the definition of regularity it follows that the vectors $\varphi'(0), \ldots, \varphi^{(d-1)}(0)$ are linearly independent. Since the solution curve $\varphi(t)$ is contained into the invariant hyperplane π, we obtain that π coincides with the linear space

$$\varphi(0) + \langle \varphi'(0), \ldots, \varphi^{(d-1)}(0) \rangle. \tag{8}$$

Here, as usual $\langle \varphi'(0), \ldots, \varphi^{(d-1)}(0) \rangle$ denotes the vector space generated by the vectors $\varphi'(0), \ldots, \varphi^{(d-1)}(0)$.

Moreover, since π is invariant, from (8) it follows that the vector $\varphi^{(d)}(0)$ depends linearly of $\varphi'(0), \ldots, \varphi^{(d-1)}(0)$. Therefore, from (6) we obtain that the determinant $\rho(A_d)$ of the matrix

$$\begin{pmatrix} P_1 & \cdots & P_d \\ DP_1 & \cdots & DP_d \\ \vdots & \ddots & \vdots \\ D^{d-1}P_1 & \cdots & D^{d-1}P_d \end{pmatrix},$$

evaluated at $\varphi(0)$ vanishes. Since the degree of the polynomials P_i, DP_i, $D^2P_i, \ldots, D^{d-1}P_i$ are at most $m_i, m_i + m_1 - 1, \ldots, m_i + (d-1)m_1 - (d-1)$ respectively, we obtain that the degree of the above determinant is at most

$$N = m_1 + \cdots + m_d + (m_1 - 1)\binom{d}{2}.$$

Hence the coordinate A_d of $\varphi(0)$ must be a real root of the polynomial $\rho(A_d)$ of degree at most N.

Now we claim that for each A_d we can determine in a unique way all coefficients $A_1, \ldots, A_{d-1}$ of the hyperplane (7). Since the solution curve $\varphi(t) = (\varphi_1(t), \ldots, \varphi_d(t))$ is contained in the invariant hyperplane π, we have that

$$A_1\varphi_1(t) + \cdots + A_{d-1}\varphi_{d-1}(t) - \varphi_d(t) + A_d = 0,$$

for all t in the open interval I. Therefore

$$A_1\varphi_1^{(k)}(t) + \cdots + A_{d-1}\varphi_{d-1}^{(k)}(t) - \varphi_d^{(k)}(t) = 0,$$

for $k = 1, 2, \ldots$. Hence, from (8) we obtain that the vector $(A_1, \ldots, A_{d-1}, -1)$ is ortogonal to π.

From (6) and the definition of B_P it follows easily that the vector $B_P(\varphi(0))$ is ortogonal to the vectors $\varphi'(0), \ldots, \varphi^{(d-1)}(0)$ in $\mathbf{R}^d$. So $(A_1, \ldots, A_{d-1}, -1)$ and $B_P(\varphi(0)) = B_P(0, \ldots, 0, A_d)$ are parallel. Hence we can compute $A_1, \ldots, A_{d-1}$ in function of A_d in a unique way. Consequently the claim follows, and the proposition is proved. $\square$

A lower bound of the maximal number of invariant hyperplanes that a regular polynomial system of degree $\mathbf{m}$ in $\mathbf{R}^d$ can have (i.e., a lower bound for $\alpha(d, \mathbf{m})$) will be given in the next proposition. First we need some preliminary notations and definitions.

The map $g:\{1,\dots,d\}\to\mathbf{N}$ given by $g(k)=m_k$ is monotone non-increasing, because $m_i\geqslant m_j$ if $i<j$. Then g defines a partition of $\{1,\dots,d\}$ into subsets $I_i=[\alpha_i,\dots,\beta_i]$ with $i=1,\dots,s$ such that $\alpha_i=\beta_{i-1}+1$ for $i=2,\dots,s$, g is constant over each I_i, and $g(I_i)>g(I_j)$ if $i<j$. Let l_i be the cardinality of the set I_i. Eventually we can have that $\alpha_i=\beta_i$, and consequently $l_i=1$.

For each $n\in\mathbf{N}$, let $p(n)$ equal to 0 if n is even, and equal to 1 if n is odd. Then, we define

$$\gamma(d;\mathbf{m})=\left(\sum_{i=1}^{d}m_i\right)+\sum_{\substack{j=1\\ l_j\neq 1}}^{s}\bigl(1+p(m_{\alpha_j})\bigr)\binom{l_j}{2}.$$

PROPOSITION 7. – *Let* $P_i=F_i(x_i)=\prod_{k=1}^{m_i}(x_i-k)$ *for* $i=1,\dots,d$ (*eventually* m_d *can be zero, with* P_d *equal to a constant different from zero*). *We assume that* $0\leqslant m_s\leqslant 1$ *implies* $m_{s-1}>m_s$. *Then system* (1) *is regular and has exactly* $\gamma(d;\mathbf{m})$ *invariant hyperplanes.*

In order to prove Proposition 7, we need the following lemma.

LEMMA 8. – *Under the assumptions of Proposition* 7 *suppose that there exist integers* r,k *such that* $1<r\leqslant k\leqslant d$ *and*

$$m_1\geqslant\cdots\geqslant m_{r-1}>m_r=\cdots=m_k.$$

Then, the invariant hyperplanes which can be written in the form

$$x_k=\sum_{j=1}^{k-1}A_jx_j+A\quad\text{with}\ \sum_{j=1}^{k-1}|A_j|\neq 0,$$

are

(a) $x_k=x_r,\dots,x_k=x_{k-1}$ *if* m_k *is even.*
(b) $x_k=x_r,\dots,x_k=x_{k-1}$ *and* $x_k=-x_r+m_k+1,\dots,x_k=-x_{k-1}+m_k+1$ *if* m_k *is odd.*

Proof. – The hyperplanes of the statement of the lemma are invariant by the flow of the system of Proposition 7 if and only if

$$\prod_{i=1}^{m_k}\left(\sum_{j=1}^{k-1}A_jx_j+A-i\right)=\sum_{j=1}^{k-1}A_j\prod_{i=1}^{m_j}(x_j-i).\tag{9}$$

In a first step we prove that $A_1 = \cdots = A_{r-1} = 0$. In effect, for every $s \in \{1, \ldots, r-1\}$, we take $x_s = \beta$ with $\beta \in \{1, \ldots, m_s\}$ and $x_j = 0$ for $j = 1, \ldots, k-1$, $j \neq s$, in (9). Thus, we obtain that the polynomial in β of degree m_k,

$$\prod_{i=1}^{m_k} (A_s\beta + A - i) = \alpha$$

would have $m_s > m_k$ roots, where

$$\alpha = \left(\sum_{j=1}^{k-1} A_j (-1)^{m_j} m_j! \right) - A_s (-1)^{m_s} m_s!.$$

Consequently, $A_s = 0$ and in the expresion of our hyperplane only the coefficients of the variables x_j such that $m_j = m_k$ can be nonzero; i.e. $A_1 = \cdots = A_{r-1} = 0$.

In the rest of this proof we assume $m_r = \cdots = m_k = m \geqslant 2$. Taking $x_1 = \cdots = x_{k-1} = 0$ in (9) it follows that

$$a = \sum_{j=r}^{k-1} A_j = \frac{\prod_{i=1}^{m} (A - i)}{(-1)^m m!}. \tag{10}$$

We consider two cases.

Case 1: $A \notin \{1, \ldots, m\}$. Taking now $x_1 = \cdots = x_{k-1} = \beta$ with $\beta \in \{1, \ldots, m\}$ in (9) and using (10) we get that

$$\left[\prod_{i=1}^{m} (A - i) \right] \left[\prod_{j=1}^{m} \left(\frac{\beta a}{A - j} + 1 \right) \right] = 0.$$

This equality implies that for each $\beta = 1, \ldots, m$ there is a unique $j_\beta \in \{1, \ldots, m\}$ such that

$$\frac{\beta a}{A - j_\beta} + 1 = 0, \tag{11}$$

and that the map $\beta \mapsto j_\beta$ is bijective. By using (10) this equality becomes

$$\beta \prod_{i=1}^{m} (A - i) = (-1)^{m+1} m! (A - j_\beta).$$

Multiplying all these equalities for $\beta = 1, \ldots, m$ we obtain

$$\left[\prod_{i=1}^{m}(A-i)\right]^{m-1} = (-1)^{m(m+1)}(m!)^{m-1}.$$

Therefore, from (10) we get that $a^{m-1} = 1$. So, $a = 1$ if m is even, and $a = \pm 1$ if m is odd. Now we consider two subcases.

Subcase 1: $a = 1$. Then, from (11) $A = j_\beta - \beta$ for all $\beta = 1, \ldots, m$. Since $j_\beta \in \{1, \ldots, m\}$, we obtain $A = 0$.

Taking $x_j = 1$ and all other x_l equal to 0, expression (9) becomes

$$\prod_{i=1}^{m}(A_j - i) = (A_j - 1)(-1)^{m+1}m!. \tag{12}$$

Clearly $A_j = 1$ is a solution of (12). Removing this solution, expression (12) goes over to

$$\prod_{i=2}^{m}(A_j - i) = (-1)^{m+1}m!. \tag{13}$$

This equality implies immediately that all real roots of this polynomial in A_j cannot be negative; i.e., $A_j \geqslant 0$. Since this argument can be made for all $j \in \{r, \ldots, m\}$ and $a = 1$, we obtain that $0 \leqslant A_j \leqslant 1$. Therefore, from (13) A_j must be 0. In short, the unique real roots A_j (12) satisfying $a = 1$ are 0 or 1. Consequently, the values of $A_r, \ldots, A_{k-1}$ are all 0 except one of them which is 1. Hence in this subcase the unique invariant hyperplanes of the form $x_k = \sum_{j=1}^{k-1} A_j x_j + A$ are $x_k = x_j$ for $j = r, \ldots, k-1$.

Subcase 2: $a = -1$. Recall that now m is odd. Then, from (11) $A = j_\beta + \beta$ for all $\beta = 1, \ldots, m$. Since $j_\beta \in \{1, \ldots, m\}$, we obtain $A = m + 1$.

Taking $x_j = 1$ and all other x_l equal to 0, expression (9) becomes

$$\prod_{i=1}^{m}(A_j + i) = (A_j + 1)(-1)^{m+1}m!. \tag{14}$$

Using the same kind of arguments than in the previous subcase we obtain that A_j can be only -1 or 0 for all $j = r, \ldots, k-1$. Since $a = -1$, the values of $A_r, \ldots, A_{k-1}$ are all 0 except one of them which is -1.

Hence in this subcase the unique invariant hyperplanes of the form $x_k = \sum_{j=1}^{k-1} A_j x_j + A$ are $x_k = -x_j + m + 1$ for $j = r, \ldots, k-1$.

Case 2: $A \in \{1, \ldots, m\}$. Then, from (10) $a = 0$. Therefore, taking $x_j = \beta \in \{1, \ldots, m\}$ and all other x_l equal to 0, expression (9) becomes

$$\prod_{i=1}^{m} (\beta A_j + A - i) - A_j (-1)^{m+1} m! = 0.$$

This is a polynomial in β of degree m having the roots $1, 2, \ldots, m$. So the above equality can be written as

$$A_j^m \prod_{j=1}^{m} (\beta - j) = 0.$$

Consequently the independent coefficient of both polynomials must be equal, i.e.

$$(-1)^{m+1} m! A_j = (-1)^m m! A_j^m. \tag{15}$$

Clearly $A_j = 0$ is a solution of this equality. We claim that this solution is the unique solution satisfying that $a = 0$. The solutions A_j of (15) different from 0 must satisfy $A_j^{m-1} = -1$. If m is odd there is no solution. If m is even, then $A_j = -1$. In short, for all $j \in \{r, \ldots, k-1\}$ we have that $A_j \in \{-1, 0\}$. Hence, since $a = 0$, it should be $A_j = 0$ for all $j = r, \ldots, k-1$. But these hyperplanes are not consider in the lemma because we assume that $\sum_{j=1}^{k-1} |A_j| \neq 0$. □

Proof of Proposition 7. – It is immediate to check that the $m_1 + \cdots + m_d$ invariant hyperplanes

$$x_i = j, \quad i = 1, \ldots, d, \; j = 1, \ldots, m_i,$$

are the unique invariant hyperplanes parallel to some of the axes of coordinates. All other possible invariant hyperplanes can be written as

$$\begin{aligned} x_d &= A_1 x_1 + \cdots + A_{d-1} x_{d-1} + A, \\ x_{d-1} &= A_1 x_1 + \cdots + A_{d-2} x_{d-2} + A, \\ &\vdots \\ x_2 &= A_1 x_1 + A. \end{aligned}$$

We note that the above equation starting with x_k for $k = 2, \ldots, d$ corresponds to the hyperplane that has all coefficients of the variables $x_{k+1}, \ldots, x_d$ equal to zero, and the coefficients of x_k and of some of the variables $x_1, \ldots, x_{k-1}$ are nonzero.

By Lemma 8, we obtain that the number of invariant hyperplanes defined by the above expressions and starting with x_k for $k \in I_i = [\alpha_i, \beta_i]$ is $(1 + p(m_{\alpha_i}))(\beta_i - k)$. Consequently, the number of hyperplanes that correspond to the interval $I_i = [\alpha_i, \beta_i]$ is:

$$\big(1 + p(m_{\alpha_i})\big)\big((l_i - 1) + (l_i - 2) + \cdots + 1\big) = \big(1 + p(m_{\alpha_i})\big)\binom{l_i}{2}.$$

Hence the total number of invariant hyperplanes is given by $\gamma(d; \mathbf{m})$.

To end the proof of the proposition we only need to show that the vector field $P = (P_1, \ldots, P_d)$ with $P_i = F_i(x_i)$ for $i = 1, \ldots, d$ is regular. Clearly the first two conditions of regularity are satisfied by our vector fields P. In order to prove the third condition we must show that on each invariant hyperplane π of P the set of singular points of B_P is an algebraic variety of codimension at most 1 in π.

Take π equal to $\{x_i = j\}$ for $i = 1, \ldots, d$ and $j = 1, \ldots, m_i$. Since the i–th column of the matrix M_P is $(F_i(x_i), DF_i(x_i), \ldots, D^{d-2}F_i(x_i))$, this column is identically 0 on the points of π, and the other columns are independent on the x_i variable. So, from the definitions of B_l, over the points of π all $B_l = 0$ if $l \neq i$, and B_i is a polynomial in the variables $x_1, \ldots, x_{i-1}, x_i, \ldots, x_d$. Therefore, $B_i = 0$ on π is an algebraic variety of codimension 1.

Consider now π of the form $\{x_k = x_j\}$. Then the kth and jth columns of M_P are equal, and the other columns are independent of the variables x_k and x_j. So, on π we have that $B_l = 0$ if $l \notin \{k, j\}$, and B_k and B_j are equal except perhaps in the sign. As above $B_k = 0$ or $B_j = 0$ on π is an algebraic variety of codimension 1.

Finally if π is a hyperplane of the form $\{x_k = -x_j + m + 1\}$. The same arguments of the previous two cases show that $B_k = 0$ on π is an algebraic variety of codimension 1. This completes the proof of the proposition. □

THEOREM 9. – *Assume that the polynomial system* (1) *of degree* $\mathbf{m} = (m_1, \ldots, m_d)$ *with* $m_1 \geqslant \cdots \geqslant m_d$ *satisfies that* $d \geqslant 2$, $m_1 \geqslant 2$, *and that if* $0 \leqslant m_r \leqslant 1$ *then* $m_{r-1} > m_r$. *Then the following statements hold.*

(a) $\gamma(d;\mathbf{m}) \leqslant \alpha(d;\mathbf{m}) \leqslant \beta(d;\mathbf{m})$ *for all* $d \geqslant 2$.

(b) $\gamma(d;\mathbf{m}) = \beta(d;\mathbf{m})$ *for all* $d \geqslant 2$ *if and only if* $\mathbf{m} = (2, \ldots, 2)$ *or* $\mathbf{m} = (3, \ldots, 3)$.

Proof. – By Propositions 5 and 6 it follows immediately statement (a). We claim that the difference

$$\beta(d;\mathbf{m}) - \gamma(d;\mathbf{m}) = (m_1 - 1)\binom{d}{2} - \sum_{\substack{j=1 \\ l_j \neq 1}}^{s} \bigl(1 + p(m_{\alpha_j})\bigr)\binom{l_j}{2},$$

is positive if $m_1 = \cdots = m_d$ does not hold. We assume that $\{1, \ldots, d\}$ is the union of s subsets $I_1, \ldots, I_s$ such that $g(I_i) = m_i$ with $m_1 > \cdots > m_s$, $s > 2$ and cardinals $l_1, \ldots, l_s$, respectively. If $m_1 = 2$, then $s \leqslant 3$, $l_j \in \{0, 1\}$ and the claim is verified trivially. If $m_1 > 2$, since $1 + p(m_i) \leqslant 2$ and

$$\binom{d}{2} > \binom{l_1}{2} + \cdots + \binom{l_s}{2},$$

we obtain that

$$(m_1 - 1)\binom{d}{2} > \sum_{\substack{j=1 \\ l_j \neq 1}}^{s} \bigl(1 + p(m_{\alpha_j})\bigr)\binom{l_j}{2},$$

and, thus, the claim also is verified. The proof of statement (b) concludes checking that in systems (1) of constant degree (i.e., $\mathbf{m} = (m, \ldots, m)$), the equality (b) is verified if and only if $m = 2, 3$. □

OPEN PROBLEM. – *Determine the exact value of* $\alpha(d, \mathbf{m})$.

This open problem has only been solved for some values of $d = 2$ see [1], and under the assumptions of statement (b) of Theorem 9.

As usual we denote by $E(x)$ the integer part function of the real number x.

THEOREM 10. – *Under the assumptions of Theorem* 9 *the following statements hold*:

(a) *If* $\mathbf{m} \in \{(2,0),(3,0),(3,1),(3,1,0)\}$ *or* $m_1 \geqslant 4$ *except* $\mathbf{m} = (4,4), \mathbf{m} = (4,4,4)$, *then* $\Delta(d;m_1-1)+1 > \alpha(d;\mathbf{m})$.

(b) *On verify* $\Delta(d;m_1-1)+1 \leqslant \alpha(d;\mathbf{m})$, *if*

$$\mathbf{m} \in \bigl\{(2,2,0),(2,2,1,0),(3,3,3,1),(3,3,3,3,2,2),\\ (3,3,3,3,3,2,0)\bigr\}$$

or $m_1 \leqslant 3$ *and one of the following conditions is verified*:

(b.1) $m_1 = \cdots = m_d = 2$ *with* $d \geqslant 2$;
(b.2) $m_1 = \cdots = m_{d-1} = 2$ *and* $m_d = 0$ *with* $d \geqslant 4$;
(b.3) $m_1 = \cdots = m_{d-1} = 2$ *and* $m_d = 1$ *with* $d \geqslant 3$;
(b.4) $m_1 = \cdots = m_{d-2} = 2$, $m_{d-1} = 1$ *and* $m_d = 0$ *with* $d \geqslant 5$;
(b.5) $m_1 = \cdots = m_d = 3$ *with* $d \geqslant 2$;
(b.6) $m_1 = \cdots = m_{d-1} = 3$ *and* $0 \leqslant m_d \leqslant 1$ *with* $d \geqslant 5$;
(b.7) $m_1 = \cdots = m_{d-1} = 3$ *and* $m_d = 2$ *with* $d \geqslant 4$;
(b.8) $m_1 = \cdots = m_{d-2} = 3$, $m_{d-1} = 1$ *or* $m_{d-1} = 2$ *and* $m_d = 0$ *with* $d \geqslant 8$;
(b.9) $m_1 = \cdots = m_{d-2} = 3$ *and* $m_{d-1} = 2, m_d = 1$ $d \geqslant 7$;
(b.10) $m_1 = \cdots = m_{d-3} = 3$, $m_{d-2} = 2$, $m_{d-1} = 1$, $m_d = 0$ *with* $d \geqslant 11$;
(b.11) $m_1 = \cdots = m_s = 3$ *and* $m_{s+1} = \cdots = m_d = 2$ *with* $s > 4$ *and* $s+2 \leqslant d \leqslant s + E(s/2)$;
(b.12) $m_1 = \cdots = m_s = 3$, $m_{s+1} = \cdots = m_{d-1} = 2$ *and* $0 \leqslant m_d \leqslant 1$ *with* $s \geqslant 7$ *and* $s+3 \leqslant d \leqslant s + E(s/2) + p(s) - 1$;
(b.13) $m_1 = \cdots = m_s = 3$, $m_{s+1} = \cdots = m_{d-2} = 2$, $m_{d-1} = 1$ *and* $m_d = 0$ *with* $s \geqslant 10$ *and* $s+4 \leqslant d \leqslant s + E(s/2) - 1$.

Proof. – Let $D = \Delta(d;m_1-1)+1$. Then we denote by

$$J_\alpha = D - \alpha(d;\mathbf{m}), \qquad J_\beta = D - \beta(d;\mathbf{m}), \qquad J_\gamma = D - \gamma(d;\mathbf{m}).$$

Then statement (a) holds if $J_\alpha > 0$. This is easy to verify for $\mathbf{m} \in \{(2,0),(3,0),(3,1),\ (3,1,0)\}$. Finally we study the case $m_1 \geqslant 4$. We separate the proof into two subcases: $m_1 = 4$ and $m_1 > 4$.

Case $m_1 = 4$ and $\mathbf{m} \notin \{(4,4),(4,4,4)\}$. As

$$J_\beta = \frac{1}{6}\bigl[(d+3)(d+2)(d+1) - 9d^2 + 9d - 18\bigr] - (m_2 + \cdots + m_d),$$

it is easy to obtain that $J_\beta > 0$ if $d = 4, 5$ and also for $d = 2, 3$ when $\mathbf{m}$ is such that $m_d \neq 4$. For $d \geqslant 6$,

$$J_\beta \geqslant \frac{1}{6}\left[9(d+2)(d+1) - 9d^2 + 9d - 18\right] - (m_2 + \cdots + m_d) \geqslant 2d + 4 > 0.$$

In short, using Theorem 9(a) we have that $J_\alpha \geqslant J_\beta$, and consequently the proof of statement (a) for $m_1 = 4$ follows.

Case $m_1 > 4$. We consider

$$\begin{aligned}
J_\beta(d;\mathbf{m}) &\geqslant \binom{d+m_1-1}{m_1-1} + 1 - (m_1-1)\binom{d}{2} - dm_1 \\
&= \sum_{k=0}^{m_1-1}\binom{d+k-1}{k} + 1 - (m_1-1)\binom{d}{2} - dm_1 \\
&= \left[\sum_{k=0}^{3}\binom{d+k-1}{k} + 1 - 3\binom{d}{2} - 4d\right] \\
&\quad + \sum_{k=4}^{m_1-1}\binom{d+k-1}{k} - (m_1-4)\binom{d}{2} - (m_1-4)d \\
&= J_\beta\big(d;(4,\ldots,4)\big) + \sum_{k=4}^{m_1-1}\left[\binom{d+k-1}{k} - \binom{d}{2} - \binom{d}{1}\right] \\
&= J_\beta\big(d;(4,\ldots,4)\big) + \sum_{k=4}^{m_1-1}\left[\binom{d+k-1}{k} - \binom{d+1}{2}\right].
\end{aligned}$$

Since $J_\beta(d;(4, m_2, \ldots, m_d)) \geqslant 0$ and

$$\binom{d+1}{2} < \binom{d+k-1}{k},$$

for all $k = 4, \ldots, m_1 - 1$, it follows that $J_\beta(d, \mathbf{m}) > 0$. This completes the proof of statement (a).

In order to prove statement (b), from Theorem 9(a), it is sufficient to show that $J_\gamma \leqslant 0$. The proof is organized as follows. First, we study the case $m_1 = 2$. Then, from the assumptions we must only consider the cases $(2, \ldots, 2)$, $(2, \ldots, 2, 0)$, $(2, \ldots, 2, 1)$ and $(2, \ldots, 2, 1, 0)$, where the blocks $2, \ldots, 2$ contain at least one 2.

Assume that $\mathbf{m} = (2, \ldots, 2)$. Then, $J_\gamma = -(d^2 + d - 4)/2$. Therefore, for $d > 0$ we have that $J_\gamma = 0$ if and only if $d = (\sqrt{17} - 1)/2 \in (1, 2)$. Hence, $J_\gamma < 0$ if and only if $d \geqslant 2$. So statement (b.1) is proved.

Assume that $\mathbf{m} = (2, \ldots, 2, 0)$. If $d = 2$, then $J_\gamma = 2$ and $J_\beta = 1$. So the case $d = 2$ is not contained in statement (b). If $d \geqslant 3$, then $J_\gamma = (-d^2 + d + 6)/2$. Therefore, for $d > 0$ we have that $J_\gamma = 0$ if and only if $d = 3$. Hence, $J_\gamma < 0$ if and only if $d \geqslant 4$, and consequently (b.2) is proved. If $d = 3$, then $J_\beta < 0 = J_\gamma$. So, the case $d = 3$ is not contained in statement (b).

Assume that $\mathbf{m} = (2, \ldots, 2, 1)$. If $d = 2$, then $J_\beta = 0 < J_\gamma$. So the case $d = 2$ is not contained in statement (b). If $d \geqslant 3$, then $J_\gamma = (-d^2 + d + 4)/2$. Since $J_\gamma = 0$ for $d \in (1, 3)$, it follows easily that $J_\gamma < 0$ if and only if $d \geqslant 3$. Hence, (b.3) is proved.

Assume that $\mathbf{m} = (2, \ldots, 2, 1, 0)$. If $d = 3$, then $J_\beta = -1$ and $J_\gamma = 2$. So the case $d = 3$ is not contained in statement (b). If $d = 4$, then $J_\beta < 0 = J_\gamma$. If $d > 4$, then $J_\gamma = (-d^2 + 3d + 4)/2$. Since $J_\gamma = 0$ if $d = 4$, (b.4) follows easily.

Now we must study the case $m_1 = 3$. Then, from the assumptions we can only consider the case $(3, \ldots, 3)$; the cases $(3, \ldots, 3, 0)$, $(3, \ldots, 3, 1)$, $(3, \ldots, 3, 2)$, $(3, \ldots, 3, 1, 0)$, $(3, \ldots, 3, 2, 0)$, $(3, \ldots, 3, 2, 1)$ and $(3, \ldots, 3, 2, 1, 0)$, where the blocks $3, \ldots, 3$ can be formed by one or more 3's; and the cases $(3, \ldots, 3, 2, \ldots, 2)$, $(3, \ldots, 3, 2, \ldots, 2, 0)$, $(3, \ldots, 3, 2, \ldots, 2, 1)$ and $(3, \ldots, 3, 2, \ldots, 2, 1, 0)$, where the blocks $3, \ldots, 3$ can be formed by one or more 3's and $2, \ldots, 2$ by, at least, two elements. The proof for $m_1 = 3$ is similar to the proof for $m_1 = 2$, but it is longer. This completes the proof of the theorem. □

From Theorem 2(b), it follows that for the polynomial systems satisfying the assumptions of Theorem 10(a), the number of invariant hyperplanes itself will be not sufficient for finding a first integral of the system using the Darboux theory of integrability. But this does not prevent that with a number of invariant hyperplanes smaller than $\Delta(d; m_1 - 1) + 1$ we can apply Theorem 2(a) for obtaining a first integral.

From Theorems 2(b) and 9(b), it follows that there are polynomial systems satisfying that $\Delta(d; m_1 - 1) + 1 \leqslant \alpha(d; \mathbf{m})$, and consequently a first integral of such systems can be constructed using the Darboux theory of integrability.

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